Space-Time Approach to Holonomy Scattering

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A general concept of holonomy scattering is defined. It applies, among other things, to imply a nontrivial (generally, inelastic) scattering amplitude for the scattering of pure flux tubes off one another, when these fluxes are non-Abelian and do not commute. The amplitudes for such processes, as well as the original Aharonov-Bohm cross section, are calculated directly from an expression for the propagator with definite winding number.

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(1) Introduction.—There is a peculiar fascination to processes where gauge fields which are locally trivial everywhere (except perhaps at isolated points) nonetheless give rise to observable physical effects. A paradigm of such behavior is the celebrated Aharonov-Bohm effect,[1,2] whereby a charged particle scatters from a small region threaded by magnetic flux even if the flux is confined to a region where the particle cannot penetrate, or to a region which is infinitesimally small.

In this Letter we argue that the basic mechanism responsible for this effect also operates in a considerably more general context. Perhaps the most striking effect, on which we shall focus for the sake of concreteness, is that *pure magnetic flux tubes* of non-Abelian flux generically scatter off one another, both elastically and inelastically. (The non-Abelian Aharonov-Bohm effect between a charge and a flux tube is well known.[3,4]) We shall calculate the amplitude for such processes, using a method that is easily generalized to more general electric-magnetic composites. As a by-product, we shall reproduce the original Aharonov-Bohm result in an extremely simple and direct fashion.

(2) Flux-tube holonomy.—Consider a gauge theory with gauge group G broken down by the Higgs mechanism to a discrete non-Abelian subgroup H. (Several aspects of such symmetry breaking have been discussed recently in Refs. 5–8.) For simplicity, we consider a $(2+1)$-dimensional theory. In this situation there may be stable flux tubes, characterized by the fact that parallel transport around closed paths surrounding them results in a nontrivial gauge transformation—or, as we say, holonomy—lying in H. We wish to consider the possible scattering of such flux tubes off one another. We will idealize our problem by imagining G to be so badly broken that at sufficient small energies and momenta we can regard the flux tubes as point singularities and ignore the massive gauge bosons. Any nontrivial scattering we find under these circumstances is evidently a purely quantum, purely global, and intrinsically non-Abelian effect, for the gauge fields vanish classically and are trivial locally, and there is clearly no corresponding scattering for Abelian magnetic flux tubes.

There is an essential subtlety in specifying the flux associated with a non-Abelian flux tube, i.e., the parallel transport around it. That is, this flux is not gauge invariant. If we perform a gauge transformation by $h \in H$, then the parallel transport g around the string is modified to hgh^{-1}. This would seem to indicate that the flux is defined only to within a conjugacy class. However, when we have two or more flux tubes the ambiguity is still only one overall conjugation. Thus it seems appropriate, and it is certainly convenient, to fix a gauge and to speak as if each flux tube is characterized by a definite group element. Of course all physical results must then be invariant against an overall conjugation.

Now we come to the heart of the matter: Why do such flux tubes scatter at all? The reason is most easily grasped visually. We refer to Fig. 1, where the passage of one flux-tube-charge composite (a,ξ) over another (b,η) is depicted. From this figure, it is apparent that the effect of this passage is to make the change

$$\begin{aligned}(a',\xi') &= (b,\eta)\,,\\ (b',\eta') &= (bab^{-1},R(b)\xi)\,,\end{aligned} \tag{2.1}$$

where the charges transform according to the representation R. Iterating this once, we find the result for a complete winding of one composite particle around another:

$$\begin{aligned}(a^{(1)},\xi^{(1)}) &= (bab^{-1},R(b)\xi)\\ &= ((ba)a(ba)^{-1},R(ba)R(a)^{-1}\xi)\,,\\ (b^{(1)},\eta^{(1)}) &= ((bab^{-1})^{-1}b(bab^{-1}),R(bab^{-1})\xi)\\ &= ((ba)b(ba)^{-1},R(ba)R(b)^{-1}\eta)\,.\end{aligned} \tag{2.2}$$

In the second step of each of these equations we have written them in such a way that the result of further iterations can be read off easily. Noting the $b^{(1)}a^{(1)} = ba$, we readily find that after p windings

$$\begin{aligned}(a^{(p)},\xi^{(p)}) &= ((ba)^p a(ba)^{-p},R(ba)^p R(a)^{-p}\xi)\,,\\ (b^{(p)},\eta^{(p)}) &= ((ba)^p b(ba)^{-p},R(ba)^p R(b)^{-p}\eta)\,.\end{aligned} \tag{2.3}$$

For pure flux tubes, a result essentially equivalent to (2.2) was given by Carlip[9] in the context of $(2+1)$-dimensional gravity. He phrased it in terms of a geometric construction known as the Dehn twist. It may

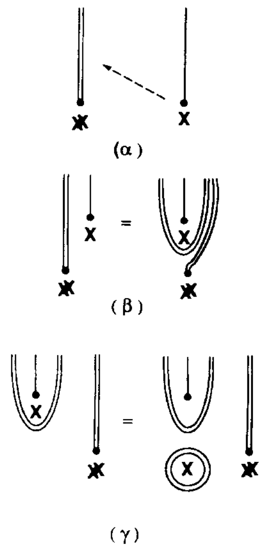

FIG. 1. The modification of charge-flux composites by passage around one another. The composites are represented by a charge, located at the cross or double cross, and a singular line across which there is nontrivial parallel transport. Note that these two structural elements must be slightly separated, to give an unambiguous specification of the quantum numbers. As one composite winds counterclockwise through another —operation α—it hits the singular line; to pull it through the singular line, we add a tube and antitube to the right and perform operation β. The resulting object on the left is not quite of the form of a charge split from a flux tube; to put it in that form we perform operation γ.

be appropriate to add a word concerning the relationship between these approaches.

To investigate the dynamics of our flux tubes, we must consider how to weight the amplitudes contributed by different possible space-time trajectories in a Feynman path integral. Since the gauge field outside the tubes is locally trivial, the very most naive reaction is to ignore it; but of course Aharonov and Bohm[1] taught us long ago that this is wrong. The next level of sophistication is to realize that the global residue of flux outside the tubes may be localized, by appropriate gauge transformations, onto small lines emanating from the tubes (i.e., it is only within these regions that the vector potential and parallel transport are nontrivial). Now as we consider a trajectory where flux tubes wind around one another we can, by appropriate gauge transformations, change the direction of the lines emanating from them in such a way that they never cross one another. In this way, it seems that once again we have defined away the interaction. However, the result of shunting the singular lines around this way is that they get tangled up, as shown in Fig. 2. A test charge circling just one or the other of our composite particles in the final state will, in this description, have to cross several singular lines. In general, therefore, it will be parallel transported through a different transformation than for the original unwound configuration, which

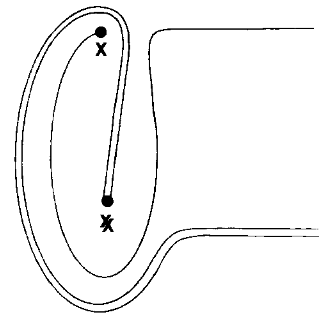

FIG. 2. An alternative procedure to Fig. 1, where the singular lines are twisted to avoid one another as one charge-flux composite winds around the other.

involved just crossing a single line. Similarly, a test flux winding around one of our composite particles will be affected by its passage through the surrounding spaghetti. Thus the effective fluxes and charges in the final state are definitely different from their original values in this (Carlip's) description as well. In fact it is not difficult to convince oneself that both descriptions of the process lead to the same result, although in our opinion the first is more transparent.

Although (2.2) and (2.3) are quite pretty and symmetric, there is one apparent asymmetry in them, which may appear puzzling. That is, ba appears in a preferred role—but what distinguishes this order, as opposed to ab? The origin of this asymmetry can be traced to the fact that we chose to implement the winding by passing the rightmost particle over the leftmost, instead of the leftmost under the rightmost. Of course these two procedures must be equivalent, and one quickly recognizes that they are related by a gauge transformation on the final state; indeed it is easy to check that conjugation by $bab^{-1}a^{-1}=(ba)(ab)^{-1}$ takes us from the first to the second.

(3) Winding amplitudes and scattering.—We adopt a space-time approach using a path integral in polar coordinates (r,θ). For simplicity, assume the target is infinitely massive and located at the origin. Consider the amplitude for the particle traveling from the point (r,θ) to (r',θ') in the time interval $(0,T)$. It is a sum over contributions of paths with different windings.[10,11] The contribution of paths with winding number n [defined by $\Theta\equiv\int_0^T dt\,\dot\theta(t)=\theta'-\theta+2\pi n$)] is the free amplitude[12,13]

$$K^{(n)}_{\text{free}}(\mathbf{x}',\mathbf{x};T)=\int d\lambda\, e^{i\lambda(\theta'-\theta+2\pi n)}Q_{|\lambda|}(r',r;T) \quad (3.1)$$

multiplied by an appropriate group factor, where

$$Q_{|m+\alpha|}(r',r;T) = \frac{\mu}{2\pi i\hbar T}\exp\left\{\frac{i\mu(r^2+r'^2)}{2\hbar T}\right\} I_{|m+\alpha|}\left(\frac{\mu r r'}{i\hbar T}\right). \quad (3.2)$$

In the Abelian case (charge e against flux ϕ) the extra factor is the Aharonov-Bohm phase $\exp\{-i\alpha(\theta'-\theta$

$+2\pi n)\}$ with $\alpha = e\phi/2\pi\hbar c$. For the non-Abelian case in which a flux tube a scatters off another flux tube b, from (2.3) we know that if a goes around b n times in the counterclockwise sense, both a and b get conjugated by ba n times. This change under conjugation is the group-theoretical factor we mentioned before. Since H is a finite group, there exists a minimal integer p such that after p rounds the flux-tube holonomies return to the original ones: Namely,

$$a^{(p)} \equiv (ba)^p a (ba)^{-p} = a\,, \qquad b^{(p)} \equiv (ba)^p b (ba)^{-p} = b\,. \tag{3.3}$$

[Note that p must be a factor of the order of the element ba in H, and also that either equation in (3.3) implies the other.] This means that $K^{(n)}$ and $K^{(n+mp)}$ must be multiplied by the same group factor, and in total there are p possible group factors for the scattered flux tube a:

$$a^{(0)} \equiv a, \quad a^{(1)} \equiv (ba)a(ba)^{-1}, \ldots, a^{(p-1)} \equiv (ba)^{p-1} a (ba)^{1-p}\,. \tag{3.4}$$

Therefore, assuming the initial a, the final state is given by $\sum_n c_n K^{(n)}_{\text{free}} a^{(n)}$, in the form of an element in the group algebra of H.

Clearly it is more convenient to use a basis in which the action of conjugation by ba is diagonalized. In the basis $a^{(k)}$, $k=0,1,\ldots,p-1$, the action of conjugation is given by the $p\times p$ matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.5}$$

The eigenvalues of C are $\omega_j = \exp\{-i2\pi j/p\}$ ($j=0, 1,\ldots,p-1$). The corresponding eigenvectors are given by

$$h^{(j)} = \sum_{k=0}^{p-1} \frac{1}{\sqrt{p}} \exp\{-i2\pi kj/p\} a^{(k)}\,. \tag{3.6}$$

In the new basis $h^{(k)}$, the conjugation becomes diagonalized with the phases $\omega_k = \exp\{-i2\pi k/p\}$.

Thus in each channel labeled by $h^{(k)}$, the scattering looks like an Abelian Aharonov-Bohm scattering with $\alpha = k/p$. For the latter, the total propagator (with mass μ) is given by

$$K(\mathbf{x}',\mathbf{x};T) = \sum_n \int d\Theta \int d\lambda\, \delta(\theta'-\theta-\Theta+2\pi n) e^{i(\lambda-\alpha)\Theta} Q_{|\lambda|}(r',r;T) = \sum_m e^{im(\theta'-\theta)} Q_{|m+\alpha|}(r',r;T)\,. \tag{3.7}$$

Suppose both r and r' are very large. Then asymptotically

$$I_{|m+\alpha|}\left(\frac{\mu r r'}{i\hbar T}\right) \sim \left(\frac{1}{2\pi}\frac{\hbar T}{\mu r r'}\right)^{1/2} \{e^{-i|m+\alpha|\pi - i\pi/4} e^{i\mu r r'/\hbar T} + e^{+i\pi/4} e^{-i\mu r r'/\hbar T}\}\,. \tag{3.8}$$

The free-particle case is given by setting $\alpha = 0$. So

$$K(\mathbf{x}',\mathbf{x};T) - K_{\text{free}}(\mathbf{x}',\mathbf{x};T) \sim \frac{\mu e^{-i\pi/4}}{2\pi i\hbar T} \exp\left\{\frac{i\mu(r+r')^2}{\hbar T}\right\} \left(\frac{1}{2\pi}\frac{\hbar T}{\mu r r'}\right)^{1/2} S(\theta',\theta)\,, \tag{3.9}$$

with

$$S(\theta',\theta) = \sum_{m=-\infty}^{+\infty} e^{im(\theta'-\theta)} (e^{-i|m+\alpha|\pi} - e^{-i|m|\pi}) = (-i) e^{-i(\theta'-\theta)/2} \frac{\sin(\pi\alpha)}{\cos[(\theta'-\theta)/2]}\,. \tag{3.10}$$

To extract the cross section, one may proceed as follows. The probability that the particle arrives at the point $\mathbf{x}'$ is

$$\frac{P(\mathbf{x}')}{(\text{unit vol.})} = |K - K_{\text{free}}|^2 = \left(\frac{\mu}{2\pi\hbar T}\right)^2 \frac{1}{2\pi}\frac{\hbar T}{\mu r r'} |S(\theta',\theta)|^2\,. \tag{3.11}$$

Compare it with the probability that a free particle arrives at the point $\mathbf{x}_0'$ which is on the straight line connecting $\mathbf{x}$ and 0 and has the same distance as $\mathbf{x}'$ from 0:

$$\frac{P(\mathbf{x}_0')}{(\text{unit vol.})} = |K_{\text{free}}(\mathbf{x}_0',\mathbf{x};T)|^2 = \left(\frac{\mu}{2\pi\hbar T}\right)^2. \tag{3.12}$$

This leads to the cross section[14] (per unit length in the z direction)

$$\frac{d\sigma}{d\varphi} = \frac{rr'}{r+r'}\frac{P(\mathbf{x}')}{P(\mathbf{x}_0')} = \frac{1}{2\pi}\frac{\hbar T}{\mu(r'+r)} |S(\theta',\theta)|^2\,. \tag{3.13}$$

Note that the velocity of the particle is $v = (r+r')/T$, so $\mu v/\hbar$ is the wave number k. Thus,

$$\frac{d\sigma}{d\varphi} = \frac{1}{2\pi k} |S(\theta',\theta)|^2 = \frac{1}{2\pi k}\frac{\sin^2(\pi\alpha)}{\sin^2(\varphi/2)}\,, \tag{3.14}$$

where the scattering angle $\varphi = \theta'$ when $\theta = \pi$.

For the non-Abelian flux-tube–flux-tube scattering, it is of interest to go back from the $h^{(k)}$ basis to the $a^{(k)}$

basis. We obtain the scattering amplitude in the channel $a^{(k)} \to a^{(j)}$,

$$S_{jk}(\theta',\theta) = (-i)\frac{e^{-i(\theta'-\theta)/2}}{\cos[(\theta'-\theta)/2]}\sum_{l=0}^{p-1}\sin\frac{l\pi}{p}b_{jl}^{*}b_{lk}\,, \quad (3.15)$$

where $b_{jk} \equiv (1/\sqrt{p})\exp\{i2\pi kj/p\}$ is the inverse of the coefficient matrix in (3.6). Putting everything together, the cross section for the channel $a^{(k)} \to a^{(j)}$ is given by

$$\frac{d\sigma}{d\varphi}(k \to j) = \frac{1}{2\pi k}\frac{|A_{jk}|^2}{\sin^2(\varphi/2)}\,, \quad (3.16)$$

with

$$A_{jk} = \left\{\sin\frac{\pi}{p}\right\}\left\{2\sin\frac{(j-k+\frac{1}{2})\pi}{p}\sin\frac{(j-k-\frac{1}{2})\pi}{p}\right\}^{-1}. \quad (3.17)$$

Note that A_{jk} and, therefore, the scattering depends only on the integer p, but not on other detailed properties of a and b. Also j and k appear only in the combination $j-k$.

(4) Comments.— We close with a number of comments.

(i) The core of the calculation depends only upon the fact that in some basis the amplitude for scattering with some number of windings is modulated by a phase proportional to the number of windings. Thus it would be straightforward to calculate the scattering of objects carrying both non-Abelian flux and charge: One need only diagonalize the operation in (2.2).

(ii) If the particles being scattered are regarded as indistinguishable, we should include the exchange process. For the case of Aharonov-Bohm scattering, this was done in Ref. 15. Again, the problem resolves itself into a group-theoretic part, which here involves diagonalizing (2.1), and a dynamic part which is essentially the same as for the Abelian case.

(iii) In our opinion non-Abelian charge-flux-tube composites are the proper generalization of anyons[16,17] to the non-Abelian case. They arise as quasiparticles in a simple generalization of the flux-phase construction, as follows. The essence of the flux-phase construction is the postulation of a correlation

$$\Gamma_{ij} = \langle c_i^\dagger c_j \rangle \quad (4.1)$$

for hopping between sites i and j, such that the total amplitude for hopping around a closed loop contains a complex phase depending on the area and the orientation (and no other properties) of the loop. With such correlations, the system has manufactured an effective magnetic field. If there are several types of fermions—e.g., with different spin, or coming from different components of a disconnected Fermi surface—then, in this construction, the same phase is assigned to each. Generically, quasiparticles around such an ordered state will be associated with defects in the ordering—vortices—and will carry fractional statistics. Now if we generalize (4.1) by allowing Γ to depend on indices labeling fermion types, then plausibly the quasiparticles acquire non-Abelian statistics. Though we certainly cannot claim at present that ordering of this type occurs in the ground state of any realistic two-dimensional Hamiltonian, such a possibility does not seem absurd and deserves further investigation.

(iv) Although we have been able to get an exact result for the two-body scattering problem, things appear to get very difficult when one considers any more complicated problems, say even scattering from two fixed centers.

(v) Verlinde[18] has been considering scattering problems for particles interacting with a Chern-Simons gauge field. This appears to be closely related to, but not identical with, the problem we have considered here, which arises for the totally broken phase of an ordinary gauge theory.

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